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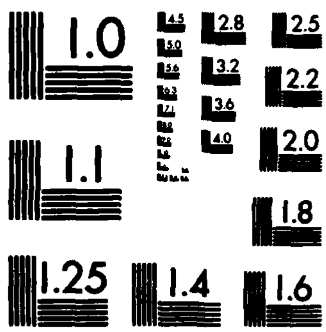

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OSCILLATORY INSTABILITY IN A TWO-FLUID BÉNARD PROBLEM

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Yuriko Renardy[1] and Daniel D. Joseph*,[2]

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ABSTRACT

A linear stability analysis for a two-layer Bénard problem is considered. The equations are not self-adjoint. The system can lose stability to time-periodic disturbances. For example, it is shown numerically that when the viscosities and coefficients of cubical expansion of the fluids are different, a Hopf bifurcation can occur, resulting in a pair of travelling waves or a standing wave. This may have application in the modelling of convection in the Earth's mantle.

AMS (MOS) Subject Classifications: 76E15, 76E20, 76T05, 76V05

Key Words: Overstability, Bénard instability, Two-component flow, Convective instability

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SIGNIFICANCE AND EXPLANATION

Flows involving two incompressible viscous fluids exhibit nonuniqueness in the sense that many interface positions are allowed when their densities are equal. Two-fluid flows also have quite different dynamical features from one-fluid flows. The one-fluid Bénard problem in which the fluid, lying between parallel horizontal plates, is heated from below has a static solution for which a linear stability analysis yields no complex eigenvalues. In this paper we show that when two fluids are involved, the arrangement in horizontal layers can have complex eigenvalues at criticality and therefore can sustain disturbances which are oscillatory in time. This may have application to the theory of convection in the Earth's mantle, which is sometimes based on the assumption that convection takes place in chemically uniform layers.

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OSCILLATORY INSTABILITY IN A TWO-FLUID BÉNARD PROBLEM

Yuriko Renardy[1] and Daniel D. Joseph*,[2]

1. Introduction

The flow of two immiscible fluids often exhibits phenomena which are without parallel in the flow of one fluid. An example is the steady shear flow of two fluids with different viscosities but similar densities. Such flows are described in Joseph, Nguyen and Beavers[1]. In this paper, we consider the Bénard problem with two fluids lying between infinite parallel plates, heated from below, and we look for new phenomena.

In the Bénard problem for one fluid, the 'exchange of stabilities' holds and all the eigenvalues of the linearized problem are real. In the two-fluid problem, we have both a real and a complex spectrum.

Busse[2] noted that convection in a two-fluid Bénard problem heated from below can admit solutions where the fluids lie in layers as well as solutions in which there are convection cells of one fluid surrounded by streamlines of the second fluid. We examine the linear stability of the arrangement where the fluids lie in two layers with a flat horizontal interface. Zeren and Reynolds[3] considered this problem, including the effect of a linear temperature gradient on the surface tension (Marangoni effect). They state that they do not know if there are purely imaginary eigenvalues at criticality. They note that Sterling and Scriven[4] found purely imaginary eigenvalues in the problem where the upper fluid is inviscid and the convection is induced by surface tension, which depends linearly on the temperature of the free surface (Marangoni problem). Zeren and Reynolds

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chose to compute neutral stability curves corresponding to zero eigenvalues. We concentrate on the Bénard problem without the Marangoni effect and show that the equations are not self-adjoint. We give an example of a situation when the marginal eigenvalues are a purely imaginary conjugate pair of multiplicity 2 (the same eigenvalues appear for negative wavenumbers). Marginal eigenvalues of this type are associated with Hopf bifurcations from the motionless state to either a pair of travelling waves or a standing wave (Ruelle[5]). According to Ruelle[5], both the travelling and standing waves are solutions to the nonlinear problem. If they are both supercritical, then only one of them can be stable; otherwise, they are both unstable. The possibility of travelling waves on the interface of immiscible fluids may have application to the modelling of mantle convection (Busse[2]).

2. Linear Stability Analysis

We consider the linear stability of a two-dimensional (x^*, z^*) problem when the bottom fluid (fluid 1) occupies a layer from $z^* = 0$ to $z^* = \ell_1^*$ and the top fluid (fluid 2) lies between $z^* = \ell_1^*$ and $z^* = \ell^*$. Asterisks denote dimensional variables. The plate at $z^* = 0$ is at temperature $T_0^* + \Delta T^*$, $\Delta T^* > 0$, and the plate at $z^* = \ell^*$ is at temperature T_0^*. Fluid i (i = 1,2) has a coefficient of cubical expansion $\hat{\alpha}_i$, thermal diffusivity κ_i, thermal conductivity k_i, viscosity μ_i, kinematic viscosity ν_i and density ρ_i at temperature T_0^*. We define a Rayleigh number $R = g\hat{\alpha}_1 \Delta T^* \ell^{*3}/(\kappa_1 \nu_1)$, a Prandtl number $Pr = \nu_1/\kappa_1$ and a surface tension parameter $Tn = S\ell^*/(\kappa_1 \mu_1)$, where S is the surface tension, all based on fluid 1. There are 6 dimensionless ratios:

$m = \mu_1/\mu_2$, $r = \rho_1/\rho_2$, $\gamma = \kappa_1/\kappa_2$, $\zeta = k_1/k_2$, $\beta = \hat{\alpha}_1/\hat{\alpha}_2$ and $\ell_1 = \ell_1^*/\ell^*$. Denote $\ell_2 = 1 - \ell_1$.

We choose the following dimensionless variables (without asterisks): $(x,z) = (x^*,z^*)/\ell^*$, $t = \kappa_1 t^*/\ell^{*2}$, $\underline{u} = \underline{u}^* \ell^*/\kappa_1$, $T = T^*/\Delta T^*$, $p = p^* \ell^{*2}/\rho_1 \kappa_1^2$ where $\underline{u}^*$ is the velocity (u^*, w^*), p^* is the pressure and T^* is the temperature. The unperturbed temperature is

$$T = \begin{cases} T_0 + 1 - A_1 z & \text{for } 0 < z < \ell_1 \\ T_0 + A_2(1 - z) & \text{for } \ell_1 < z < 1 \end{cases} \tag{1}$$

where $A_1 = \dfrac{1}{\ell_1 + \zeta \ell_2}$ and $A_2 = \zeta A_1$ and the unperturbed motion is static. A linear perturbation proportional to $\exp(\sigma t + i\alpha x)$ is superposed on the velocity, temperature and interface position.

The perturbation θ to the temperature satisfies

$$\begin{aligned} \sigma\theta - wA_1 &= \nabla^2\theta, \quad \text{for } 0 < z < \ell_1 , \\ \sigma\theta - wA_2 &= \frac{1}{\gamma}\nabla^2\theta, \quad \text{for } \ell_1 < z < 1 . \end{aligned} \tag{2}$$

We use the Boussinesq approximation in the Navier-Stokes equations. Hence, the density in the buoyancy term is approximated by

$$\rho_i(1 - \hat{\alpha}_i(T^* - T_0^*)), \quad i = 1,2 \tag{3}$$

to yield:

$$
\begin{aligned}
\sigma \underline{u} &= -\nabla p + RPr\, \theta \underline{e}_z + Pr\, \nabla^2 \underline{u} \quad \text{for} \quad 0 < z < \ell_1 \\
\sigma \underline{u} &= -r\nabla p + \frac{RPr}{\beta}\, \theta \underline{e}_z + \frac{r}{m}\, Pr\, \nabla^2 \underline{u} \quad \text{for} \quad \ell_1 < z < 1
\end{aligned}
\tag{4}
$$

where $\underline{e}_z$ is the unit vertical vector. Incompressibility yields

$$\nabla . \underline{u} = 0 \ . \tag{5}$$

The boundary conditions are: $\underline{u} = 0$, $\theta = 0$ at $z = 0,1$. The following linearized interface conditions (see Zeren and Reynolds[3] for complete derivation) hold at $z = \ell_1$. $[\cdot]$ denotes $\cdot_1 - \cdot_2$. Continuity of velocity, shear stress, temperature and heat flux are, respectively,

$$
\begin{gathered}
[w] = [\partial w/\partial z] = 0 \\
[\mu(\partial^2 w/\partial z^2 + \alpha^2 w)] = 0 \\
[\theta] = h[A] \\
[k\partial\theta/\partial z] = 0 \ .
\end{gathered}
\tag{6}
$$

The kinematic free-surface condition is

$$w = \sigma h \tag{7}$$

where the perturbed free-surface position is $z = \ell_1 + h(x,t)$ and

$$h = h_0 \exp(i\alpha x + \sigma t) \ . \tag{8}$$

The conservation of volume of the incompressible fluids implies that the $h(x,t)$ has a zero mean value as a function of x. This is automatic if $\alpha \neq 0$. There is a difference between $\alpha = 0$ and $\alpha \to 0$, the former is disallowed. The balance of normal stress is

$$
\begin{aligned}
&\frac{1}{m}\, \partial^3 w_2/\partial z^3 - \partial^3 w_1/\partial z^3 + 3\alpha^2 (1 - \frac{1}{m}) \partial w_1/\partial z \\
&- h\alpha^2 (R\{\frac{(\frac{1}{r} - 1)}{\hat{\alpha}_1 \Delta T} + \ell_2 A_2 (1 - \frac{1}{r\beta})\} - \alpha^2 . Tn) = \frac{\sigma}{Pr} (\frac{1}{r} - 1) \partial w_1/\partial z \ .
\end{aligned}
\tag{9}
$$

We will show that the above problem is not self-adjoint. Hence, the eigenvalues need not be real. Let Ω be a strip of width one wavelength $2\pi/\alpha$, covering $0 < z < 1$. Let Ω_1 be the part of Ω in fluid 1 and Ω_2 be in fluid 2. Let $\bar{\underline{u}}^*$ and $\bar{\theta}^*$ be the

complex conjugates of the adjoints of $\underline{u}$ and θ. The asterisks here denote the adjoint and the overbars the complex conjugates. Integration by parts of

$$\int_{\Omega_1} \bar{\underline{u}}^* \cdot (\sigma\underline{u} + \nabla p - \Pr \nabla^2\underline{u} - RPr\theta\underline{e}_z)$$

$$+ \int_{\Omega_2} \bar{\underline{u}}^* \cdot (\frac{\sigma}{r}\underline{u} + \nabla p - \frac{1}{m}\Pr \nabla^2\underline{u} - \frac{RPr}{r\beta}\theta\underline{e}_z) \tag{10}$$

$$+ \int_{\Omega_1} \bar{\theta}^*(\sigma\theta - wA - \nabla^2\theta) + \int_{\Omega_2} \frac{\bar{\theta}^*}{\zeta}(\gamma[\sigma\theta - wA] - \nabla^2\theta)$$

yields

$$\int_{\Omega_1} \underline{u} \cdot (\sigma\bar{\underline{u}}^* - \Pr[\nabla^2\bar{\underline{u}}^* + \nabla(\nabla.\bar{\underline{u}}^*)] - A\bar{\theta}^*\underline{e}_z) - \int_{\Omega} p\nabla.\bar{\underline{u}}^*$$

$$+ \int_{\Omega_2} \underline{u} \cdot (\frac{\sigma}{r}\bar{\underline{u}}^* - \frac{\Pr}{m}[\nabla^2\bar{\underline{u}}^* + \nabla(\nabla.\bar{\underline{u}}^*)] - \frac{A\gamma}{\zeta}\bar{\theta}^*\underline{e}_z) \tag{11}$$

$$+ \int_{\Omega_1} \theta(\sigma\bar{\theta}^* - RPr\bar{w}^* - \nabla^2\bar{\theta}^*) + \int_{\Omega_2} \theta(\frac{\gamma\sigma}{\zeta}\bar{\theta}^* - \frac{RPr}{r\beta}\bar{w}^* - \frac{1}{\zeta}\nabla^2\bar{\theta}^*) - B$$

where B consists of boundary integrals taken at $z = \ell_1$ over one wavelength in x. We give the expression for B later. The above integration is facilitated by expressing $\int_{\Omega_1} \nabla^2\underline{u} \cdot \bar{\underline{u}}^*$ as $\int_{\Omega_1} \bar{\underline{u}}^* \cdot \nabla.(\nabla\underline{u} + (\nabla\underline{u})^T)$, where superscript T denotes the transpose, taking advantage of $\nabla.\underline{u} = 0$, to obtain, for example,

$$\int_{\Omega_1} \nabla^2\underline{u} \cdot \bar{\underline{u}}^* = \int_{\Omega_1} \underline{u} \cdot \nabla^2\bar{\underline{u}}^* + (\underline{u} \cdot \nabla)(\nabla.\bar{\underline{u}}^*)$$

$$+ \int_{\substack{x=0 \\ (z=\ell_1)}}^{\frac{2\pi}{\alpha}} [\bar{u}^*(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) + 2\bar{w}^*\frac{\partial w}{\partial z} - u(\frac{\partial \bar{w}^*}{\partial z} + \frac{\partial \bar{w}^*}{\partial x}) - 2w\frac{\partial \bar{w}^*}{\partial z}]dx \; .$$

Choosing $\underline{u}$, θ, p and its derivatives to vanish in the neighbourhood of the interface, we find $\nabla \cdot \bar{\underline{u}}^* = 0$ and other adjoint equations. Since $\nabla \cdot \underline{u} = 0$, the coefficients of $\underline{u}$ in (11) do not vanish but are the gradients of a function we denote by $\bar{p}^*$. Hence,

$$\sigma\bar{\underline{u}}^* - Pr\nabla^2\bar{\underline{u}}^* - A\bar{\theta}^*\underline{e}_z = -\nabla\bar{p}^* ,$$

in fluid 1 (12)

$$\sigma\bar{\theta}^* - RPr\bar{\underline{w}}^* - \nabla^2\bar{\theta}^* = 0$$

and

$$\frac{\sigma}{r}\,\bar{\underline{u}}^* - \frac{Pr}{m}\,\nabla^2\bar{\underline{u}}^* - \frac{\gamma A}{\zeta}\,\bar{\theta}^*\underline{e}_z = -\nabla\bar{p}^* ,$$

in fluid 2 (13)

$$\frac{\gamma}{\zeta}\,\sigma\bar{\theta}^* - \frac{RPr}{r\beta}\,\bar{w}^* - \frac{1}{\zeta}\,\nabla^2\bar{\theta}^* = 0 .$$

We examine B to find the adjoint interface conditions. The integration is over one wavelength in x at $z = \ell_1$.

$$B = \int - [p\bar{w}^*] + Pr\Big[\bar{u}_1^*\Big(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x}\Big) - \frac{\bar{u}_2^*}{m}\Big(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\Big)$$

$$+ \frac{u_2}{m}\Big(\frac{\partial\bar{u}_2^*}{\partial z} + \frac{\partial\bar{w}_2^*}{\partial x}\Big) - u_1\Big(\frac{\partial\bar{u}_1^*}{\partial z} + \frac{\partial\bar{w}_1^*}{\partial x}\Big) + 2\bar{w}_1^*\frac{\partial w_1}{\partial z} - \frac{2}{m}\bar{w}_2^*\frac{\partial w_2}{\partial z}$$

$$- 2w_1\frac{\partial\bar{w}_1^*}{\partial z} + \frac{2}{m}w_2\frac{\partial\bar{w}_2^*}{\partial z}\Big] + \bar{\theta}_1^*\frac{\partial\theta_1}{\partial z} - \frac{\bar{\theta}_2^*}{\zeta}\frac{\partial\theta_2}{\partial z} + \frac{\theta_2}{\zeta}\frac{\partial\bar{\theta}_2^*}{\partial z} - \theta_1\frac{\partial\bar{\theta}_1^*}{\partial z}\,dx$$

$$= \int - \Big(p_1 - 2Pr\frac{\partial w_1}{\partial z}\Big)\bar{w}_1^* + \Big(p_2 - \frac{2Pr}{m}\frac{\partial w_2}{\partial z}\Big)\bar{w}_2^* - 2Pr\Big(w_1\frac{\partial\bar{w}_1^*}{\partial z} - \frac{w_2}{m}\frac{\partial\bar{w}_2^*}{\partial z}\Big)$$

$$+ Pr\Big[[\bar{u}^*]\Big(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x}\Big) + u_1\Big(\frac{1}{m}\Big(\frac{\partial\bar{u}_2^*}{\partial z} + \frac{\partial\bar{w}_2^*}{\partial x}\Big) - \Big(\frac{\partial\bar{u}_1^*}{\partial z} + \frac{\partial\bar{w}_1^*}{\partial x}\Big)\Big)\Big]$$

$$+ [\bar{\theta}^*]\frac{\partial\theta_1}{\partial z} - \Big[\theta_1\frac{\partial\bar{\theta}_1^*}{\partial z} - \frac{\theta_2}{\zeta}\frac{\partial\bar{\theta}_2^*}{\partial z}\Big]dx .$$

Conditions (6) and (7) yield

$$[\underline{u}^*] = [\bar{\theta}^*] = [\mu(\frac{\partial \bar{u}^*}{\partial z} + \frac{\partial \bar{w}^*}{\partial x})] = 0 \ . \tag{14}$$

We can add $\int_\Omega \underline{u} \cdot \nabla \bar{p}^*$ to (10) which introduces $\int [p^* w]dx$ into B in (11). Hence,

$$B = \int - \bar{w}_1^*(p_1 - 2Pr \frac{\partial w_1}{\partial z}) + \bar{w}_2^*(p_2 - \frac{2Pr}{m} \frac{\partial w_2}{\partial z}) + w_1(\bar{p}_1^* - 2Pr \frac{\partial \bar{w}_1^*}{\partial z})$$

$$- w_2(\bar{p}_2^* - \frac{2Pr}{m} \frac{\partial \bar{w}_2^*}{\partial z}) - Pr[\theta_1 \frac{\partial \bar{\theta}_1^*}{\partial z} - \frac{\theta_2}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z}]dx \ .$$

Condition (9) can be written as

$$p_2 - p_1 + 2Pr(\frac{\partial w_1}{\partial z} - \frac{1}{m} \frac{\partial w_2}{\partial z}) + h[M_1 + \alpha^2 M_2] = 0 \tag{15}$$

where

$$M_1 = RPr[\frac{(1 - \frac{1}{r})}{\hat{\alpha}_1 \Delta T} + \frac{1}{r\beta} - 1] \quad \text{and} \quad M_2 = PrTn \ .$$

Using (6) and $-\alpha^2 \equiv \frac{\partial^2}{\partial x^2} \equiv {}''$, the last term in (15) is

$$\frac{1}{[A]} \left[[\theta]M_1 - [\theta'']M_2 \right] . \tag{16}$$

We use (6), (15) and (16) to obtain

$$B = \int w_1[(\bar{p}_1^* - 2Pr \frac{\partial \bar{w}_1^*}{\partial z}) - (\bar{p}_2^* - \frac{2Pr}{m} \frac{\partial \bar{w}_2^*}{\partial z})]$$

$$- [\bar{w}^*]\left[p_1 - 2Pr \frac{\partial w_1}{\partial z} - \theta_1 \frac{M_1}{[A]} + \theta_1'' \frac{M_2}{[A]}\right]$$

$$- \frac{1}{[A]} [\bar{w}^*(\theta M_1 - \theta'' M_2)] - \theta_1 \frac{\partial \bar{\theta}_1^*}{\partial z} + \frac{\theta_2}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} dx \ .$$

We choose

$$[\bar{w}^*] = 0 \tag{17}$$

and use (7) to find

$$B = \int \frac{[\theta]}{[A]} \Big[\sigma\Big([\bar{p}_1^*] - 2Pr \frac{\partial \bar{w}_1^*}{\partial z} + \frac{2Pr}{m} \frac{\partial \bar{w}_2^*}{\partial z}\Big) - \Big(\bar{w}_1^* M_1 - \bar{w}_1^{*\prime\prime} M_2\Big)$$

$$- [A] \frac{\partial \bar{\theta}_1^*}{\partial z} \Big] - \theta_1\Big(\frac{\partial \bar{\theta}_1^*}{\partial z} - \frac{1}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z}\Big) dx \ .$$

We choose

$$\frac{\partial \bar{\theta}_1^*}{\partial z} - \frac{1}{\zeta} \frac{\partial \bar{\theta}_2^*}{\partial z} = 0 \tag{18}$$

and

$$\sigma\Big([\bar{p}^*] - 2Pr \frac{\partial \bar{w}_1^*}{\partial z} + \frac{2Pr}{m} \frac{\partial \bar{w}_2^*}{\partial z}\Big) - \Big(\bar{w}_1^* M_1 - \bar{w}_1^{*\prime\prime} M_2\Big) - [A] \frac{\partial \bar{\theta}_1^*}{\partial z} = 0 \ . \tag{19}$$

Equations (12)-(14) and (17)-(19) are the adjoint equations.

3. Numerical Scheme

We use (5) to eliminate u so that in each fluid, we have the heat equation and one momentum equation, linear in σ:

$$\left.\begin{aligned} Pr(L^2w - \alpha^2R\theta) &= \sigma Lw, \\ wA_1 + L\theta &= \sigma\theta \end{aligned}\right\} \quad \text{for} \quad 0 < z < \ell_1 ,$$

and

$$\left.\begin{aligned} Pr(\frac{r}{m} L^2w - \frac{\alpha^2R}{\beta}\theta) &= \sigma Lw, \\ wA_2 + \frac{1}{\gamma} L\theta &= \sigma\theta \end{aligned}\right\} \quad \text{for} \quad \ell_1 < z < 1$$

where $L \equiv \partial^2/\partial z^2 - \alpha^2$. We change the variable z to z_i in fluid i defined by $z_1 = \frac{2}{\ell_1} z - 1$ and $z_2 = \frac{2}{\ell_2}(z - 1) + 1$ so that the z_i range over $[-1,1]$ in each fluid. We then expand $w(z_i)$ and $\theta(z_i)$ in powers of Chebyschev polynomials $T_m(z_i)$ (Orszag[6]) for $m = 0,\ldots,N$ giving a total of $4N + 4$ unknown coefficients. Together with the free-surface variable h_0, there are $4N + 5$ unknowns. There are 6 boundary conditions and 7 interface conditions. The term of highest differential order in the momentum equation is $\partial^4 w/\partial z^4$. Since we choose w to be an Nth degree polynomial, the term $\partial^4 w/\partial z^4$ is of degree $N - 4$ and therefore the momentum equation is truncated at the $N - 4$th degree, yielding $N - 3$ equations in each fluid. Similarly, since the term of highest differential order in the heat equation is $\partial^2 w/\partial z^2$, we truncate this equation at the $N - 2$th degree, yielding $N - 1$ equations in each fluid. The eigenvalues of the resulting $4N + 5$ square matrix equation were computed in complex double precision on a VAX11-780 using the IMSL routine EIGZC.

To check the accuracy and convergence of our computer code, we computed the eigenvalues for the Bénard problem in one fluid with $Pr = 1$, $R = 2177.41$ and 47005.6, $\alpha = 2$. The eigenvalues for this problem are real and are given by Reid and Harris[7]. The eigenvalues at criticality (at which the real part of σ should vanish) are less than 10^{-5} when $N = 15$. A convergence test with $N = 15$ and 20 showed that several other eigenvalues had converged to at least 5 figures at $N = 15$.

The computations for two fluids were checked against Zeren and Reynolds[3] by adding an extra term into the shear stress balance at the interface in order to take into account the Marangoni effect. We define a Marangoni number based on fluid 1:

$Ma = \left(-\frac{dS}{dT}\right) \frac{\Delta T^* \ell^*}{\mu_1 \kappa_1}$ and our shear stress condition at $z = \ell_1$ is modified to:

$$\alpha^2(m - 1)w_1 + m\partial^2 w_1/\partial z^2 - \partial^2 w_2/\partial z^2 + Ma.m.\alpha^2(\theta_1 - A_1 h) = 0 .$$

We used their Table 2 for the values of the physical variables at 16°C for benzene lying above water. We checked our eigenvalues against their Table 3 for $\ell_1 = 0.1$ and 0.6 for heating from below. Note that our definition of the R and Ma are different from theirs. At $\ell_1 = 0.1$, converting their parameters to ours, they find criticality at $Ma = 1255.71$, $R = 178.3045$, $\hat{\alpha}_1 \Delta T^* = 0.00032537$, $Pr = 8.1$, $\alpha = 3.5$ and $Tn = 460320$. We computed $(\sigma/Pr) = 0.006186$ using both $N = 15$ and 20. This yields 0.00175 for the eigenvalue q of Zeren and Reynolds. At $\ell_1 = 0.6$, their parameters in Table 3 become $Ma = 4016.7153$, $R = 570.3736$, $\hat{\alpha}_1 \Delta T^* = 0.0010408$ and $\alpha = 2.5$. We computed $-.00436$ for their eigenvalue q at $N = 15$ and 20. In both cases, we also found stable complex conjugate pairs in the spectrum.

4. Numerical Results

To aid the reader in the interpretation of the numerical results, we recall some results from the Bénard problem with one fluid. In the simplest case, the layer is bounded at $z = 0$ and $z = 1$ by stress-free conducting boundaries and[8]

$$\sigma = -\frac{1}{2}(1 + Pr)(n^2\pi^2 + \alpha^2) \pm \left\{\frac{1}{4}(Pr - 1)^2(n^2\pi^2 + \alpha^2)^2 + \frac{\alpha^2 R Pr}{n^2\pi^2 + \alpha^2}\right\}^{1/2} , \tag{20}$$

for $n = 1,2,\ldots$.

Hence, for $\alpha = 0$,

$$\sigma = -\frac{1}{2}(1 + Pr)n^2\pi^2 \pm \frac{1}{2}|Pr - 1|n^2\pi^2 < 0 , \tag{21}$$

and as $\alpha \to \infty$,

$$\sigma \sim -\frac{\alpha^2}{2}((1 + Pr) \pm |Pr - 1|) < 0 . \tag{22}$$

In the critical case $\sigma = 0$, the least value of R occurs when $\alpha = \frac{\pi}{\sqrt{2}}$ and $R = (\pi^2 + \alpha^2)^3/\alpha^2$. These formulas are for stress-free surfaces but they give an idea of the variation of $\sigma(R,\alpha^2)$ in the classical case of one fluid between rigid boundaries.

Now we consider the case when there are two fluids with equal properties. This would at first thought appear to be a one-fluid problem. However, it is easy to see that there is a solution with $[\theta] = [A] = 0$, $h_0 \neq 0$ and $\sigma = 0$. We shall use the nomenclature introduced by Yih[9] in a related problem and call this mode, which is important when the properties of the two fluids are different, an 'interfacial mode'. We track eigenvalues as we vary parameters. Besides the interfacial eigenfunctions, we have other eigenfunctions which we shall call Bénard modes.

In tracking the eigenvalues, we shall fix all the parameters so that there is a critical α such that $\mathrm{Re}\,\sigma(\alpha,R,Pr,Tn,m,r,\gamma,\zeta,\beta,\ell_1) = 0$ with $\mathrm{Re}\,\sigma < 0$ for other α. We shall exhibit parameters for which $\mathrm{Im}\,\sigma \neq 0$ at criticality. Hence we obtain oscillations in the linear problem at criticality ('exchange of stabilities' does not hold) and the nonlinear problem for Bénard convection in two fluids can have time-periodic solutions near criticality.

Let the two fluids have equal densities at temperature T_0^* and the same thermal diffusivities and conductivities: $r = \gamma = \zeta = 1$. We let $R = 1695.7$, $Pr = 1$, $\hat{\alpha}_1 \Delta T^* = 0.001$, $Tn = 0$, $Ma = 0$, $m = 1.1$ and $\beta = 0.9$. Thus, if fluid 1 occupies the entire flow, the Rayleigh number is lower than the critical one 1708 (see Reid and Harris[7]). If fluid 2 occupies the entire flow, R is 2072.52 and the flow is linearly unstable for a range of α. We choose $\ell_1 = 0.4$. Figure 1 is a graph of the growth rate $\mathrm{Re}\,\sigma$ against α.

We are approximately at criticality when $\alpha = 3.1$. In this case, we compute $\sigma = 0.000072 \pm i15.9259$ with $N = 15, 20$.

The five numbers next to the curves in Figure 1 denote branches which display different features. The interfacial mode is associated with branches 1, 3 and 5. Branch 1 can be obtained from the interfacial mode with $\alpha \to 0$ when the properties of the fluids are equal $(\beta = m = 1)$ by moving β to 0.9 and m to 1.1. This branch is real-valued. Branch 2 is associated with the least stable of the Bénard modes for a single fluid when $\alpha \to 0$. This branch is approximately -9.87 at $\alpha \to 0$ and would correspond to the largest value of (21). Branch 2 is real-valued. Branches 1 and 2 coalesce and split into conjugate pairs at $\alpha = 1.275$. At $\alpha = 6.79$, the conjugate pair again splits into the two real-valued branches 4 and 5. Branch 4 is associated with a Bénard mode and remains real, decreasing rapidly as α is increased, as in the single fluid problem (see (22)).

Branch 5 is an interfacial mode. It is real-valued and negative. The stability for large α which is associated with branches 4 and 5 is explained by our choice of β and the Boussinesq approximation (3). We consider the densities $\rho_i(1 - \hat{\alpha}_i(T - T_0))$ at the unperturbed interface $z = \ell_1$ when the temperature $T - T_0$ is given by (1). Then with $r = 1$ and $\beta = 0.9$, we find that $\rho_2(1 - 0.6\hat{\alpha}_2)$ is the density of fluid 2 at $\ell_1 = 0.4$ and $\rho_2(1 - 0.54\hat{\alpha}_2)$ is the density of fluid 1. Hence the heavy fluid is below and gravity may be expected to stabilize short (large α) waves. The interfacial eigenvalue on branch 5 is discussed in §5.

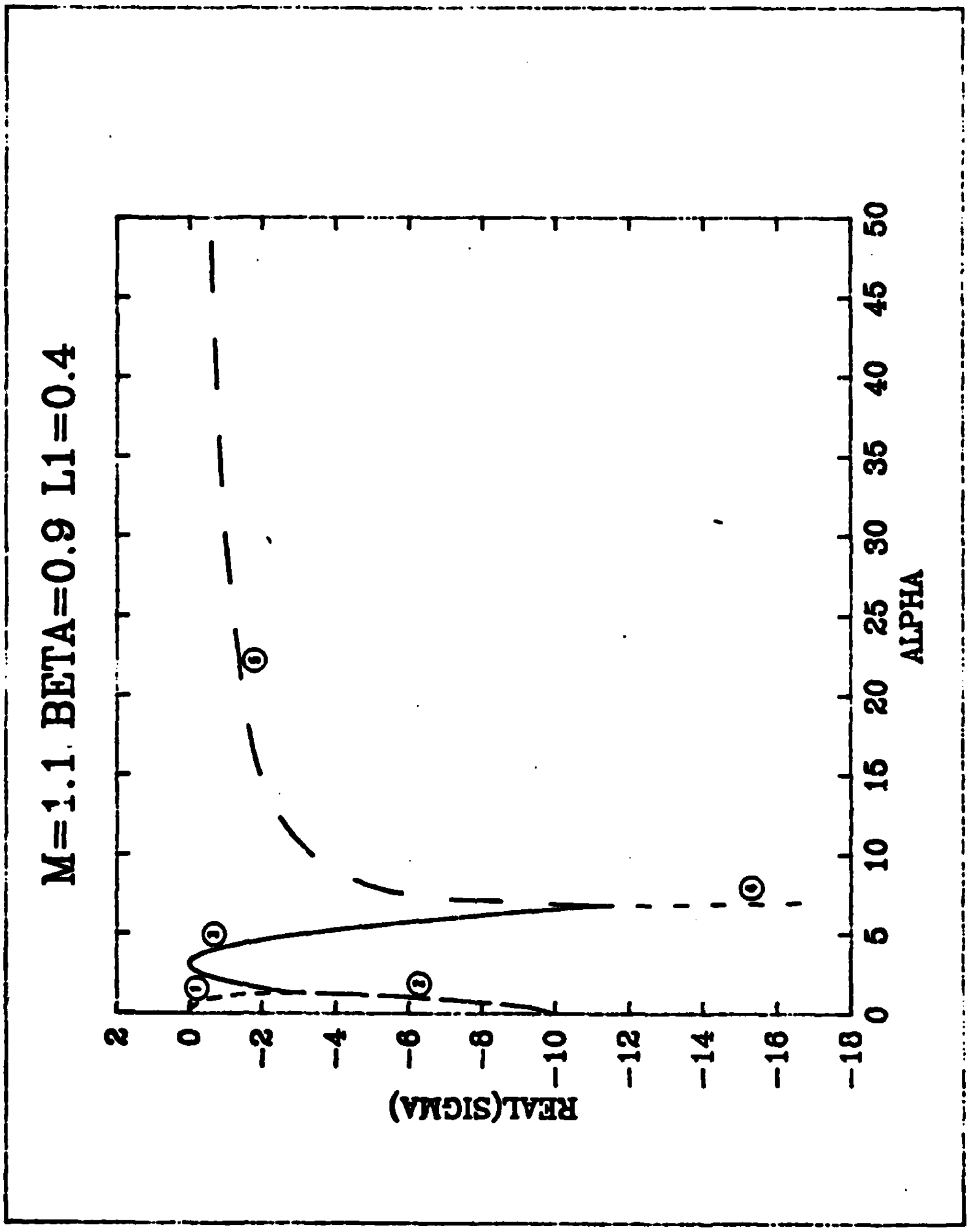

Figure 1: The growth rate $Re(\sigma)$ is plotted against the wavenumber α for $R = 1695.7$, $Pr = 1$, $\hat{\alpha}_1 \Delta T^* = 0.001$, $Tn = Ma = 0$, $r = \gamma = \zeta = 1$. Branches 1, 2, 4 and 5 belong to real-valued eigenvalues. Branch 3 consists of a complex conjugate pair.

Figure 2 shows the growth rates versus α when the Rayleigh number is increased to $R = 2177.41$ while other parameters are fixed as in Figure 1 and shows that the instability on branch 3 is associated with the complex conjugate pair of branch 3 in Figure 1.

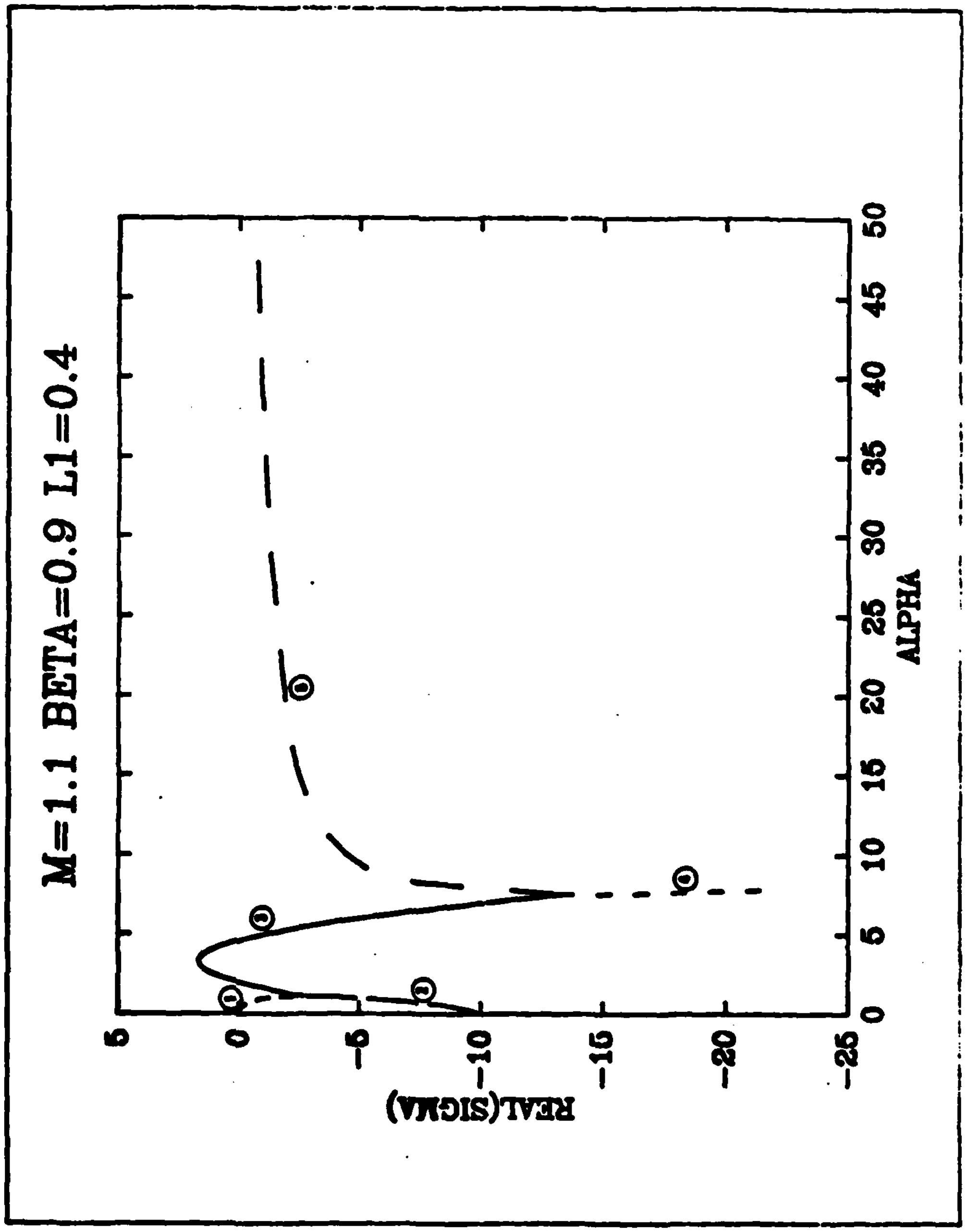

Figure 2: The growth rate $Re(\sigma)$ versus wavenumber α for $R = 2177.41$, $Pr = 1$, $\hat{\alpha}_1 \Delta T^* = 0.001$, $Tn = Ma = 0$, $r = \gamma = \zeta = 1$. Branches 1, 2, 4 and 5 belong to real-valued eigenvalues. Branch 3 consists of a complex conjugate pair.

§5. Asymptotic Analysis of the Interfacial Eigenvalue for Short Waves

We now consider disturbances of rapid variation whose length scale of variation is of the same order $O(1/\alpha)$ as the short perturbation wavelength[10].

We rescale z to $\eta = \alpha(z - \ell_1)$ and let η be $O(1)$. The equations in fluid 1 are $\sigma\theta - wA_1 = \alpha^2 L^* \theta$ where $L^* \equiv \partial^2/\partial\eta^2 - 1$ and $(\sigma - Pr\alpha^2 L^*)L^* w = -RPr\theta$. In fluid 2, $\sigma\theta - wA_2 = \frac{1}{\gamma}\alpha^2 L^* \theta$ and $(\sigma - \frac{r}{m} Pr\alpha^2 L^*)L^* w = -\frac{RPr}{\beta}\theta$. The interface conditions are:

$$w_1 = w_2 = \sigma h , \tag{23}$$

$$[\partial w_1/\partial\eta] = 0 , \tag{24}$$

$$[\theta] = [A]h , \tag{25}$$

$$m\partial^2 w_1/\partial\eta^2 - \partial^2 w_2/\partial\eta^2 + (m - 1)w_1 = 0 , \tag{26}$$

$$\zeta\partial\theta_1/\partial\eta = \partial\theta_2/\partial\eta , \tag{27}$$

$$\alpha^2(\frac{1}{m}\partial^3 w_2/\partial\eta^3 - \partial^3 w_1/\partial\eta^3) + \partial w_1/\partial\eta . 3\alpha^2(1 - \frac{1}{m}) + h\alpha^3 Tn$$

$$- hR\alpha\left[\frac{(\frac{1}{r} - 1)}{\hat{\alpha}_1 \Delta T} + \ell_2 A_2(1 - \frac{1}{r\beta})\right] = (\frac{1}{r} - 1)\frac{\sigma}{Pr}\frac{\partial w_1}{\partial\eta} . \tag{28}$$

Since the normal stress condition (28) contains both odd and even powers of α, all the variables are formally expanded in powers of $1/\alpha$. To the 0th and 1st orders, $L^*\theta = 0$ and $L^* w = 0$ in each fluid. Using conditions (23), (24) and (26), we obtain $w_1 = C_0(1 - \eta)e^{\eta} + O(1/\alpha)$ and $w_2 = C_0(1 + \eta)e^{-\eta} + O(1/\alpha)$ as $\alpha \to \infty$, which yields to this and the next order, $\partial w/\partial\eta = 0$ at the interface. Hence, the normal stress condition is

$$\alpha(\frac{1}{m}\partial^3 w_2/\partial\eta^3 - \partial^3 w_1/\partial\eta^3) - \frac{w}{\sigma} R\left[\frac{(\frac{1}{r} - 1)}{\hat{\alpha}_1 \Delta T} + \ell_2 A_2(1 - \frac{1}{r\beta})\right] = 0$$

where, for the moment, surface tension has been neglected. To avoid the trivial solution, we choose $\sigma = \sigma_0/\alpha + O(1/\alpha^2)$ for large α. The normal stress condition yields

$$\sigma_0 = \frac{R}{2(\frac{1}{m} + 1)}\left(\frac{(\frac{1}{r} - 1)}{\hat{\alpha}_1 \Delta T} + \ell_2 A_2(1 - \frac{1}{r\beta})\right) .$$

In the computations for Figure 1, the asymptotic formula is accurate to 1% for $\alpha > 20$. We computed -1.494 for $\alpha = 20$ using $N = 15$ whereas the asymptotic formula yields -1.48.

Turning now to a consideration of the effect of surface tension, we find that when $\alpha^2 Tn/R = O(1)$, then

$$\sigma_0 = \frac{R}{2(\frac{1}{m} + 1)} \left(\frac{(\frac{1}{r} - 1)}{\alpha_1 \Delta T} + \ell_2 A_2 (1 - \frac{1}{r\beta}) - \frac{\alpha^2 Tn}{R} \right) . \tag{29}$$

We computed the eigenvalue for the parameters of Figure 1 at $T = 1$ and $\alpha = 20$ to be -6.86 and the asymptotic formula yields -6.72. Equation (29) shows that surface tension is always stabilizing for short wave disturbances. The stabilization of short-waves by surface tension, even with adverse density ratios, has been found in other flows such as steady shear flows with two immiscible fluids of different viscosities[10,11].

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A linear stability analysis for a two-layer Bénard problem is considered. The equations are not self-adjoint. The system can lose stability to time-periodic disturbances. For example, it is shown numerically that when the viscosities and coefficients of cubical expansion of the fluids are different, a Hopf bifurcation can occur, resulting in a pair of travelling waves or a standing wave. This may have application in the modelling of convection in the Earth's mantle.

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